J. Fluid Mech. (2004), *vol.* 508, *pp.* 165–174.
DOI: 10.1017/S0022112004009218 Printed in the United Kingdom

Inertial levitation

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(Received 6 October 2003 and in revised form 22 March 2004)

We consider the steady levitation of a rigid plate on a thin air cushion with prescribed injection velocity. This injection velocity is assumed to be much larger than that in a conventional Prandtl boundary layer, so that inertial effects dominate. After applying the classical 'blowhard' theory of Cole & Aroesty (1968) to the two-dimensional version of the problem, it is shown that in three dimensions the flow may be foliated into streamline surfaces using Lagrangian variables. An example is given of how this may be exploited to solve the three-dimensional problem when the injection pressure distribution is known.

1. Introduction

Several solidification processes arising in the glass industry rely on the idea of floating a glass workpiece above a rigid porous medium through which air is injected at a suitably high pressure. Depending on the geometry and the flow parameters, several fluid mechanics regimes are possible; the airflow may be dominated by either inertial or viscous forces and this flow may couple with the glass flow when the workpiece has sufficiently low viscosity. Even when viscous forces are not dominant, there are several theories available to describe the mechanics of steady floating. Our starting point is the 'blowhard' theory of boundary layers with strong injection (Cole & Aroesty 1968). This theory, which is described below, considers boundary layer injection that is so strong that the transverse velocity greatly exceeds that in a traditional Prandtl boundary layer. For problems in which the flow is constrained to lie in a thin layer between two boundaries, one of which is porous, a theory can be constructed in which inertial and viscous forces are in balance, as in a traditional boundary layer (see for example Petit 1986; Hinch & Lemaître 1994; Cox 2002). However, analytical progress is possible only in geometries that are symmetric enough for similarity solutions to be obtained. In the viscous-dominated limit, these solutions tend to those of classical lubrication theory and in the inertia-dominated limit, they will be seen to agree with the theory presented below.

In this note, we will describe some models for the basic dynamics of the airflow under the assumptions that inertial effects dominate, that air compressibility is negligible and that the workpiece can be assumed to be a rigid or flexible solid plate levitating a small distance above a horizontal porous base. Even with these simplifications, the flow modelling and analysis offer some interesting theoretical challenges. As shown in § 2, these may be easily overcome in the two-dimensional case; in § 3 we will describe how some progress may be made in three-dimensional floating despite the complexity of the governing system of hyperbolic equations. The remainder of the introduction is devoted to setting up the models and reviewing some background theory.

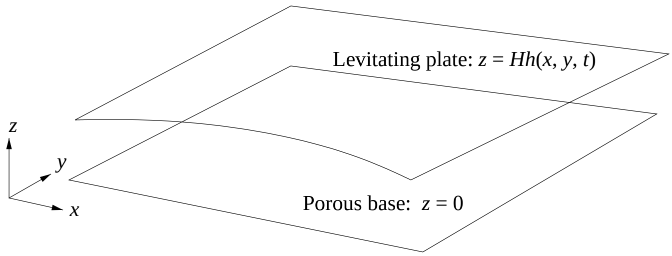

FIGURE 1. Schematic diagram of the geometry and glass sheet.

As shown in figure 1, the surface of the porous base is taken to be $z = 0$ and we assume that an injection velocity is generated by a pumping mechanism that delivers air at a known flow rate that depends on x and y with no feedback from the pressure variation in the (x, y)-plane. Neglecting viscous, compressible and gravitational forces, the steady air velocity between the base and the workpiece is (u, v, w), where

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1.1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{1.2}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{1.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{1.4}$$

Here, the constant air density is denoted by ρ and p denotes pressure. We further assume that the base is perforated in such a way that the air emerges normally, so that

$$u = v = 0, \qquad w = Ww_0(x, y)$$

at $z = 0$, where W measures a typical injection velocity.

In writing (1.1)–(1.3), we have made the crucial assumption that W is larger than the transverse velocity in the Prandtl boundary layer that will be generated at the workpiece; this means that the flow in the levitation gas layer is inviscid to lowest order, even though the layer is thin. Finally, we assume no normal flow across the base of the workpiece $z = Hh(x, y)$ (where H is a typical 'ride height') and an ambient pressure p_0 around the periphery of the workpiece, which must be an isobar.

The floating regime is one in which $\varepsilon = H/L$ is small, where L is a typical horizontal dimension of the workpiece. An estimate of ε is obtained by balancing $\rho u^2 \sim \rho W^2\varepsilon^{-2}$ with the hydrostatic pressure $\rho_g g a$ caused by the workpiece, ρ_g being its density and a its thickness. Thus $\varepsilon^2 \sim \rho W^2/(\rho_g g a)$. We non-dimensionalize according to $x = L\bar{x}$, $y = L\bar{y}$, $z = \varepsilon L\bar{z}$, $u = (W/\varepsilon)\bar{u}$, $v = (W/\varepsilon)\bar{v}$, $w = W\bar{w}$ and $p = p_0 + \rho W^2\varepsilon^{-2}\bar{p}$, where a bar indicates a non-dimensional variable: for simplicity the bars will henceforth be dropped. To lowest order, (1.1)–(1.4) become, in these scaled variables,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}, \tag{1.5}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y}, \tag{1.6}$$

$$0 = -\frac{\partial p}{\partial z}, \tag{1.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.8}$$

with

$$u = v = 0, \quad w = w_0 \quad \text{at} \quad z = 0,$$

$$(u, v, w) \cdot \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, -1 \right) = 0 \quad \text{at} \quad z = h(x, y),$$

and $p = 0$ at the workpiece periphery. The viscous boundary layer on $z = h$ is negligibly thin when $\varepsilon \gg 1/\sqrt{R_e}$, where $R_e = \varepsilon^{-1} W L/\nu$, and ν denotes the kinematic viscosity of the air.

2. Two-dimensional flow

2.1. *The Cole & Aroesty theory*

Following the procedure in Cole & Aroesty (1968), when $v = 0$ and all variables are independent of y, we may solve the hyperbolic system

$$u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{\mathrm{d}p}{\mathrm{d}x}, \qquad \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{2.1}$$

with $w = w_0(x)$ at $z = 0$ by introducing the variable x^* which denotes the x-station at which the streamline passing through (x, z) emerges from the porous base at $z = 0$. When x^* is constant, the first equation of (2.1) may be written

$$\left(u \frac{\partial}{\partial x} + w \frac{\partial}{\partial z} \right) \left(p + \tfrac{1}{2} u^2 \right) = 0$$

and hence, when x and x^* are regarded as independent variables rather than x and z (and with a slight abuse of notation),

$$p(x) + \tfrac{1}{2} u^2(x, x^*) = p(x^*), \tag{2.2}$$

since $u = 0$ when $x = x^*$ (figure 2). Moreover, conservation of mass in the shaded region of figure 2 implies that

$$\int_{x^*}^{x} w_0(s)\, \mathrm{d}s = \int_0^z u(x, \xi)\, \mathrm{d}\xi,$$

the right-hand-side integral being taken with respect to the vertical coordinate. Hence, when z is regarded as a function of (x, x^*) along a streamline, we can differentiate with respect to x^* to find the relationship between the streamline height, p, and w_0. Thus

$$\left. \frac{\partial z}{\partial x^*} \right|_{x=const} = \frac{-w_0(x^*)}{u} = \frac{-w_0(x^*)}{\sqrt{2(p(x^*) - p(x))}}. \tag{2.3}$$

Now assuming that there is a dividing streamline emerging from $x = z = 0$ (so that $z = h(x)$ corresponds to $x^* = 0$) and since, by definition, $x = x^*$ on $z = 0$, we find that

$$h(x) = \int_0^x \frac{w_0(s)}{\sqrt{2(p(s) - p(x))}}\, \mathrm{d}s. \tag{2.4}$$

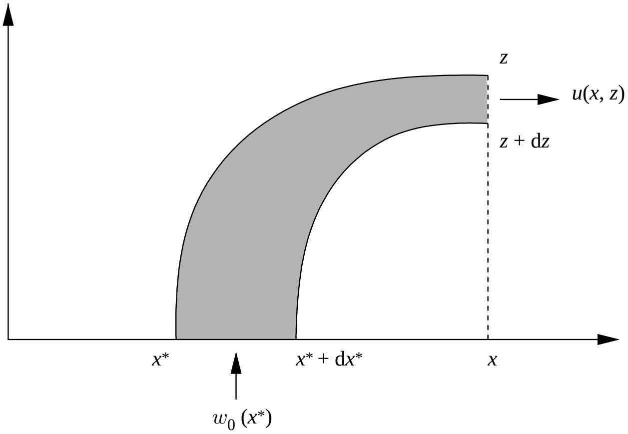

FIGURE 2. Conservation of mass for the transformation $(x, z) \rightarrow (x, x^*)$.

This generalized Abel integral equation may easily be solved for w_0 as a function of p and h, but a more practical case arises when we know w_0 and h and wish to find the pressure p. Viewed in this way, (2.4) is nonlinear. Let us suppose that $p(x)$ is a strictly monotone decreasing differentiable function with non-zero derivative over the interval $0 < x < 1$, and that $h(0)$ is finite. After the change of variables $p(s) = p(0) - S$, $p(x) = p(0) - X$ and the definitions

$$\mathscr{G}(X) = -\frac{w_0(x)}{\sqrt{2}p'(x)}, \quad \mathscr{H}(X) = h(x),$$

equation (2.4) may then be rewritten as the Abel equation

$$\mathscr{H}(X) = \int_0^X \frac{\mathscr{G}(S)}{\sqrt{X - S}}\, \mathrm{d}S. \tag{2.5}$$

Equation (2.5) may be inverted, for example by Laplace transform methods (see for example Carrier, Crook & Pearson 1966), yielding

$$\mathscr{G}(X) = \frac{\mathscr{H}(0)}{\pi\sqrt{X}} + \int_0^X \frac{\mathscr{H}'(S)}{\pi\sqrt{X - S}}\, \mathrm{d}S.$$

In general this does not seem to allow any analytical progress to be made for a general $h(x)$, as the unknown function p appears in the argument of $\mathscr{H}$. However, in the special case when $h(x)$ is identically equal to a constant h_0, the last expression can be rewritten as a differential equation for p, which is

$$\frac{-w_0(x)}{\sqrt{2}p'(x)} = \frac{h_0}{\pi\sqrt{p(0) - p(x)}},$$

with solution

$$p(x) = p(0) - \frac{\pi^2}{8h_0^2}\left(\int_0^x w_0(s)\, \mathrm{d}s\right)^2.$$

Finally, the constants $p(0)$ and h_0 are determined by the boundary condition $p(1) = 0$, and the requirement that $\int_{-1}^{1} p(x)\, \mathrm{d}x$ balances the weight of the lid. Given the density ρ_g, the thickness a of the glass sheet and the gravitational acceleration g, the weight of the workpiece per unit length in the y-direction is $2\rho_g gaL$. After non-dimensionalization, this weight is $M_G = \varepsilon^2 2\rho_g g a/\rho W^2$ so that

$$p(0) = \frac{\pi^2}{8h_0^2}\left(\int_0^1 w_0(s)\,\mathrm{d}s\right)^2, \qquad M_G = 2p(0) - \frac{\pi^2}{8h_0^2}\int_{-1}^1 \left(\int_0^x w_0(s)\,\mathrm{d}s\right)^2 \mathrm{d}x.$$

As an example, let us assume a uniform inlet velocity w_0. In this case

$$p(x) = \frac{\pi^2 w_0^2}{8h_0^2}(1-x^2),$$

and

$$h_0 = \frac{\pi w_0}{\sqrt{6M_G}}. \tag{2.6}$$

From (2.2), it then follows that

$$u(x, x^*) = \frac{\pi w_0}{2h_0}\sqrt{x^2 - x^{*2}},$$

while (2.3) leads to

$$z = h_0\left[1 - \frac{\arcsin(x^*/x)}{\pi/2}\right].$$

Hence $x^* = x\cos(\pi z/2h_0)$ and

$$u(x, z) = \frac{\pi w_0}{2h_0} x \sin\left(\frac{\pi z}{2h_0}\right).$$

Finally, integration of the continuity equation yields

$$w(x, z) = w_0 \cos\left(\frac{\pi z}{2h_0}\right),$$

thus completing the solution of (2.1). For completeness, we note from (2.6) that the dimensional ride height is

$$H = \pi\left(\frac{\rho W^2}{12\rho_g g a}\right)^{1/2} L.$$

3. Three-dimensional flow

It does not seem possible in general to isolate a closed-form equation such as (2.4) for the pressure p as a function of the floating height h. Fortunately, the inverse problem of finding h from p is more tractable, as we show in this section. Having computed $h[p(x, y)]$, one could in principle devise an iterative algorithm to invert this operation.

First, we solve the problem for the velocity components (u, v) and, for this purpose, we write equations (1.5) to (1.8) in Lagrangian variables. Let the functions

$$x = x(x^*, y^*, t), \quad y = y(x^*, y^*, t), \quad z = z(x^*, y^*, t) \tag{3.1}$$

be such that $x(x^*, y^*, 0) = x^*$, $y(x^*, y^*, 0) = y^*$, $z(x^*, y^*, 0) = 0$, and that $(u, v, w) = (\partial x/\partial t, \partial y/\partial t, \partial z/\partial t) = \partial \mathbf{x}/\partial t$, say. The momentum equations (1.5) and (1.6) become

$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial p}{\partial x}, \quad \frac{\partial^2 y}{\partial t^2} = -\frac{\partial p}{\partial y}, \tag{3.2}$$

with $\partial x/\partial t = \partial y/\partial t = 0$ at $t = 0$ and, given $p(x, y)$, they form a closed system.

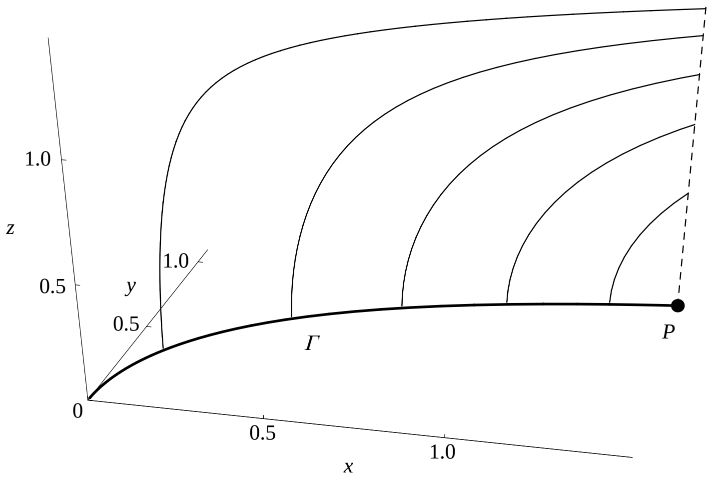

FIGURE 3. Streamlines passing above the point $P=(x, y)=(\frac{3}{2}, 1)$ for the pressure distribution (3.7) with $p_0, \kappa=\frac{1}{2}$. They emerge vertically from the locus of entry points $\Gamma(P)$.

The next step is to compute the vertical flow, which we do using conservation of mass. This can be expressed by considering the volume of an element of fluid emerging from a surface element $\mathrm{d}x^* \, \mathrm{d}y^*$ in a time $\mathrm{d}t$. This volume is initially $w_0 \mathrm{d}x^* \, \mathrm{d}y^* \, \mathrm{d}t$ and subsequently becomes $J \mathrm{d}x^* \, \mathrm{d}y^* \, \mathrm{d}t$ where $J=(\partial \boldsymbol{x}/\partial t)\cdot(\partial \boldsymbol{x}/\partial x^* \wedge \partial \boldsymbol{x}/\partial y*)$. Equating the two gives

$$\frac{\partial(x, y, z)}{\partial(x^*, y^*, t)} = w_0(x^*, y^*), \tag{3.3}$$

which, assuming (3.2) has been solved, is a linear first-order partial differential equation for the function $z(x^*, y^*, t)$, namely

$$\frac{\partial(x, y)}{\partial(y^*, t)} \frac{\partial z}{\partial x^*} + \frac{\partial(x, y)}{\partial(t, x^*)} \frac{\partial z}{\partial y^*} + \frac{\partial(x, y)}{\partial(x^*, y^*)} \frac{\partial z}{\partial t} = w_0(x^*, y^*). \tag{3.4}$$

Knowing $z(x^*, y^*, t)$, the ride height h at a point (x, y) is then found from the limiting streamline $\boldsymbol{x}(x^*, y^*, t)$ that passes tangentially to the glass sheet above that point. If we suppose that

$$\frac{\partial x}{\partial t}(0, 0, t) = \frac{\partial y}{\partial t}(0, 0, t) = 0,$$

then

$$h(x, y) = \lim_{x^*, y^* \to 0, t \to \infty} z(x^*, y^*, t),$$

in which the limit is taken with the constraint that x and y remain fixed in (3.1), i.e. we let x^* and y^* tend to the origin along the locus of entering streamlines for the point (x, y). This locus is denoted by Γ in figure 3.

The solution of (3.4) is given by solving the characteristic equations

$$\frac{\mathrm{d}x^*}{\partial(x, y)/\partial(y^*, t)} = \frac{\mathrm{d}y^*}{\partial(x, y)/\partial(t, x^*)} = \frac{\mathrm{d}t}{\partial(x, y)/\partial(x^*, y^*)} = \frac{\mathrm{d}z}{w_0(x^*, y^*)}, \tag{3.5}$$

which in fact comprise the streamlines through the points (x, y, z), where x and y are fixed and $0 < z < h(x, y)$.

We note that $h(x, y)$ is simply $\int_0^\infty (\mathrm{d}z/\mathrm{d}t)\,\mathrm{d}t$, and, denoting $\partial(x, y)/\partial(x^*, y^*)$ by $\mathscr{J}(x^*, y^*, t)$, we thus have

$$h(x, y) = \int_0^\infty \frac{w_0[x^*(x, y, t), y^*(x, y, t)]}{\mathscr{J}[x^*(x, y, t), y^*(x, y, t), t]}\,\mathrm{d}t. \tag{3.6}$$

In the Appendix, we present an alternative description of this foliation of the streamlines that offers an interesting analogy between the present flow and ray theory in optics.

3.1. *Illustrative solution*

To illustrate our point, we now consider the pressure profile

$$p = p_0(1 - x^2 - \kappa^2 y^2). \tag{3.7}$$

The isobars are elliptic and, since the pressure is ambient at the boundary, we implicitly assume an elliptic shape for the workpiece. Equations (3.2) are linear, with solution

$$x = x^* \cosh\tau, \quad y = y^* \cosh\kappa\tau,$$

where time has been rescaled using $\tau = \sqrt{2p_0}t$ for convenience. Equation (3.4) then becomes

$$-x^* \sinh\tau \cosh\kappa\tau \frac{\partial z}{\partial x^*} - \kappa y^* \sinh\kappa\tau \cosh\tau \frac{\partial z}{\partial y^*} + \cosh\tau \cosh\kappa\tau \frac{\partial z}{\partial\tau} = \frac{w_0(x^*, y^*)}{\sqrt{2p_0}},$$

and the characteristic equations are

$$\frac{-\mathrm{d}x^*}{x^* \sinh\tau \cosh\kappa\tau} = \frac{-\mathrm{d}y^*}{\kappa y^* \sinh\kappa\tau \cosh\tau} = \frac{\mathrm{d}\tau}{\cosh\tau \cosh\kappa\tau} = \frac{\sqrt{2p_0}\,\mathrm{d}z}{w_0(x^*, y^*)}.$$

Hence, on a characteristic, $x^* = x \operatorname{sech}\tau$, $y^* = y \operatorname{sech}\kappa\tau$, and

$$\frac{\mathrm{d}z}{\mathrm{d}\tau} = \frac{w_0(x^*, y^*)}{\sqrt{2p_0}} \operatorname{sech}\tau \operatorname{sech}\kappa\tau.$$

Hence,

$$z = \frac{1}{\sqrt{2p_0}} \int_0^\tau w_0(x \operatorname{sech}\tau', y \operatorname{sech}\kappa\tau') \operatorname{sech}\tau' \operatorname{sech}\kappa\tau'\,\mathrm{d}\tau',$$

and $h(x, y) = \lim_{\tau\to\infty} z$, keeping x and y fixed, i.e.

$$h(x, y) = \frac{1}{\sqrt{2p_0}} \int_0^\infty w_0(x \operatorname{sech}\tau, y \operatorname{sech}\kappa\tau) \operatorname{sech}\tau \operatorname{sech}\kappa\tau\,\mathrm{d}\tau. \tag{3.8}$$

Consequently, as in two-dimensional flows, a constant inlet velocity and a pressure profile of the form (3.7) lead to a uniform elevation $h(x, y)$.

Similarly to §2, by equating the pressure integrated over the workpiece with its weight, we find that $p_0 = 2\epsilon^2 \rho_g g a/(\rho W^2)$, so that the dimensional ride height for elliptic workpieces with area $A = \pi L^2/\kappa$ is

$$H = \left(\frac{\rho W^2 A}{4\pi\rho_g g a}\right)^{1/2} \sqrt{\kappa} \int_0^\infty \operatorname{sech}\tau \operatorname{sech}\kappa\tau\,\mathrm{d}\tau.$$

In the case $\kappa = 1$, the integral above equals one, which is consistent with the similarity solutions in Hinch & Lemaître (1994) and Cox (2002) in the zero viscosity limit.

This analysis depends crucially on the pressure having a maximum at the stagnation point. It is easy to see that if it had a saddle, then the integral in (3.8) would diverge; for example, if (3.7) is replaced by $p = p_0(1 - x^2 + \kappa^2 y^2)$, then $\mathrm{d}z/\mathrm{d}\tau = (w_0/\sqrt{2p_0})\,\mathrm{sech}\,\tau\,\sec\kappa\tau$. Indeed, if there is any direction through a stagnation point in which the pressure has a minimum, we can easily see from the dynamical system (3.2) that the Jacobian $\partial(x, y)/\partial(x^*, y^*)$ will vanish. Hence, our theory will only be acceptable under the conditions that (i) there is only one stagnation point, (ii) the pressure has a maximum there. It would be interesting to know what constraint these conditions put on the perimeter of the workpiece.

4. Concluding remarks

We have presented a theory which, in principle, describes the complicated rotational flow that occurs in three-dimensional floating, where viscous and unsteady effects are ignored. Our theory gives physically acceptable predictions only in the case where there is only one stagnation point for the horizontal velocity (u, v) and, in addition, the pressure has a maximum at that stagnation point.

Another deficiency of the theory is that, from (3.3), it cannot describe flows in which the injection velocity is zero or negative over any region of the base plate. Hence the theory cannot, as it stands, apply to injection through discrete holes in the base plate. Of course, if air were blown through a single hole of sufficient radius in the base plate, levitation could not occur because of the pressure drop as the air accelerates in the layer. In fact, it is possible to support a plate by this method by placing the base plate above the workpiece. It can also be supported in such an upside-down configuration by suction through the base plate, and this technique is also used in the glass industry. However such suction flows are not reversals of those described above.

Finally, we regret that we have been unable to present any stability analysis for our flow. Intuitively, it is likely that a horizontal plate levitated under gravity will be stable because the levitation pressure will be roughly proportional to $h^{-2}(t)$, assuming quasi-static flow. A tilting plate may be unstable in inviscid theory and, indeed, 'see-sawing' plates have been observed in practice (Hinch & Lemaître, private communication).

We would like to thank Professor E. J. Hinch for valuable comments.

Appendix

We can reformulate the two-dimensional problem (2.1) in terms of the function $z = Z(x, s)$, which denotes the vertical displacement of a streamline. The independent variable s is required to be constant on a streamline and may be chosen to be x^*. Transforming the dependent variables from (u, w, p) to (u, Z, p) and the independent variables from (x, z) to (x, s), we find that $w = u\partial Z/\partial x$, and conservation of momentum and mass now become

$$u\frac{\partial u}{\partial x} = -\frac{\mathrm{d}p}{\mathrm{d}x}, \quad \frac{\partial}{\partial x}\left(u\frac{\partial Z}{\partial s}\right) = 0,$$

with

$$Z(s, s) = 0, \qquad u(s, s) = 0, \qquad w(s, s) = w_0(s).$$

In this form, both equations may be integrated explicitly. While the former leads to the Bernoulli relation (2.2), the latter gives

$$\frac{\partial Z}{\partial s} = \frac{c(s)}{u(x, s)}, \tag{A 1}$$

where the function $c(s)$ is obtained from the boundary condition on w on $Z=0$. Noting that $(\partial Z/\partial s)(s,s) = -(\partial Z/\partial x)(s,s)$, we have $c(s) = \lim_{x\to s} u(x,s)(\partial Z/\partial s) = -w_0(s)$, which leads to equation (2.3) and the ensuing developments.

In three dimensions, let us foliate the flow into stream surfaces (surfaces composed of streamlines). To this end, we write $z = Z(x, y, s)$, where now s is any independent variable that is constant on a stream surface, so that $w = u\partial Z/\partial x + v\partial Z/\partial y$. Then (1.5)–(1.8) become respectively

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}, \qquad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}, \qquad 0 = -\frac{\partial p}{\partial s}, \tag{A 2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\log\frac{\partial Z}{\partial s} = 0, \tag{A 3}$$

and we soon find from (A 2) that $p + \frac{1}{2}(u^2+v^2)$ is constant on a streamline. It is convenient to define the curves $s = \text{const}$ in the (x, y)-plane as the level curves of p, so that

$$p(x,y) + \tfrac{1}{2}[u^2(x,y,s) + v^2(x,y,s)] = s. \tag{A 4}$$

In this way, the total head of the flow is uniform on each stream surface. As a result, the projection of the flow on the (x, y)-plane for s fixed is irrotational, for, differentiating (A 4) with respect to x and comparing it to the first equation of (A 2), we find that $\partial u/\partial y = \partial v/\partial x$. Hence, we may write

$$(u, v) = \nabla\phi(x, y, s),$$

where $\nabla \equiv (\partial/\partial x, \partial/\partial y)$. In terms of the velocity potential ϕ, the Bernoulli equation (A 4) becomes

$$\tfrac{1}{2}|\nabla\phi(x,y,s)|^2 = s - p(x,y), \tag{A 5}$$

which is the eikonal equation in optics for rays in an inhomogeneous medium. On the other hand, the continuity equation becomes

$$\nabla^2\phi + \nabla\phi\cdot\nabla\log\frac{\partial Z}{\partial s} = 0,$$

which is just the transport equation of optics, where the term $\frac{1}{2}\partial Z/\partial s$ plays the role of the field amplitude.

For a given pressure distribution and a fixed value of s, (A 5) has the characteristic equations

$$\frac{\partial x}{\partial t} = u, \quad \frac{\partial y}{\partial t} = v, \quad \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x}, \quad \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y}, \tag{A 6}$$

as in (3.2) and (3.5), where t parameterizes the characteristics, which are simply the projections of the streamlines in the (x, y)-plane. Taking r to be a coordinate along the curves $s = \text{const}$, the initial conditions for $x(r,s,t)$, $y(r,s,t)$, $u(r,s,t)$, and $v(r,s,t)$ are

$$x = x^*(r,s), \quad y = y^*(r,s), \quad u = v = 0,$$

at $t = 0$, where, as before, x^* and y^* denote the coordinates of the point of entry of the streamline through (x, y, z). On a characteristic,

$$\nabla^2\phi + \frac{\partial}{\partial t}\log\frac{\partial Z}{\partial s}(x(r,s,t), y(r,s,t), s) = 0. \tag{A 7}$$

At $t=0$, the function Z satisfies

$$Z(x(r,s,0),y(r,s,0),s)=0, \qquad \frac{\partial Z}{\partial t}(x(r,s,0),y(r,s,0),s)=w_0(x^*,y^*). \tag{A 8}$$

As in §3, by solving (A 6) for a given $p(x,y)$, we can in principle obtain x, y, u, and v as functions of r, s, and t. For fixed s, the chain rule and (A 6) combine to give

$$\nabla^2\phi=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial t}\log\frac{\partial(x,y)}{\partial(r,t)}.$$

Substituting this result in (A 7), we thus obtain

$$\frac{\partial(x,y)}{\partial(r,t)}\frac{\partial Z}{\partial s}=c(r,s),$$

which is the three-dimensional counterpart of (A 1). The function $c(r,s)$ is obtained in the same way as in the two-dimensional case by examining this equation on the plane $z=0$. We have

$$Z(x(r,s,t),y(r,s,t),s)=0 \qquad \text{at} \quad t=0.$$

Differentiating this expression once with respect to r, s, and t, successively, gives

$$\lim_{t\to 0}\frac{\partial(x,y)}{\partial(r,t)}\frac{\partial Z}{\partial s}=\lim_{t\to 0}\frac{\partial(x,y)}{\partial(s,r)}\frac{\partial Z}{\partial t}(x(r,s,t),y(r,s,t),s).$$

Therefore, using (A 8),

$$\frac{\partial(x,y)}{\partial(r,t)}\frac{\partial Z}{\partial s}=\lim_{t\to 0}\frac{\partial(x,y)}{\partial(s,r)}w_0(x^*,y^*).$$

Now, let $\Gamma(x,y)$ denote the locus of entry points on the plane $z=0$ of the streamlines that intersect the vertical axis through (x,y), as in figure 3. Then the 'ride height' of the glass workpiece is computed from

$$h(x,y)=\int_{\Gamma(x,y)}\frac{\partial Z}{\partial s}\,\mathrm{d}s,$$

which can easily be written as (3.6).

Note that for a pressure distribution with non-convex isobars, the characteristics of (A 6) intersect and form envelopes, in analogy with caustics for optical rays. The analogous optical amplitude, which is $\frac{1}{2}\partial Z/\partial s$ from (A 7), is infinite on such a caustic.

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